# All The Hydro You Never Wanted to Know

Why?

I'm starting this as a "note-to-self" as I return to reading (after a 22-year hiatus) Landau's seminal "A Hydrodynamic Theory of Multiple Formation of Particles" (1953, available as #88 in his Collected Works). This article seems more relevant than ever in light of

1) failure of the beautiful but apparently unphysical concept of "boost invariance" used by Bjorken (and essentially all subsequent workers) in his famous 1983 paper
and
2) the observation by BRAHMS of **Gaussian** *dn/dy* distributions for identified pions,
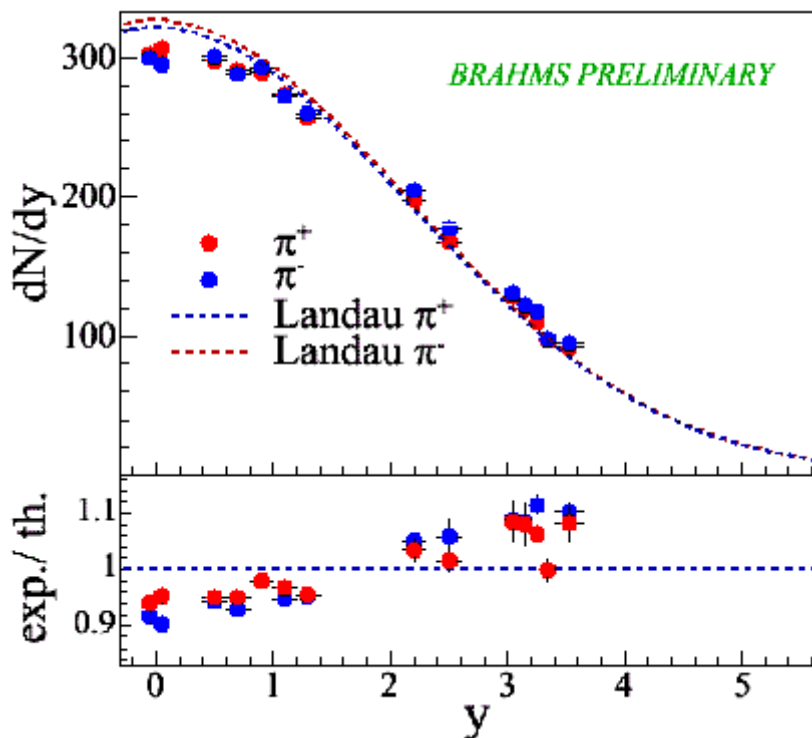
with a width consistent with Landau's 50 year old prediction(!).

So I've decided to work through pieces of it in some detail, not because I think it's the ultimate (i.e. correct) theory of RHIC collisions, but because it shows how much can be done at the "statistical" level, i.e., using only energy-momentum conservation and simple thermodynamics. (An unintended result that "appeared" is a reminder of how much harder it is to get *any* analytic results to the hydro equations of motion without the Bjorken assumption of boost invariance.) For a review of the general state-of-the-art in hydro modeling, Hirano's QM04 talk is a great place to start.

I've tried to organize the sections of these notes to have the same titles as Landau's sections. I am writing in *Mathematica* because it's convenient, and the document is best viewed by grabbing the *Mathematica* "notebook" file. In order to preserve both external and internal links when converting to PDF, I have to go through some pretty strange convolutions (see Production Notes), and in the first step of converting to HTML *Mathematica* tends to take a nice pretty notebook and make a mess, especially for equation conversion, and *especially* for "in line" equations.

Introduction

After paying due tribute ("ingenious", "very fruitful") to original Fermi statistical model, Landau immediately batters Fermi for assuming all particles are created instantly, i.e., before the strongly interacting matter begins to expand hydrodynamically. He states the condition to apply thermodynamics (a la Fermi) is the same condition required to apply hydro, i.e. $\lambda \ll L$, where λ is the mean-free-path and L is the system size (actually, Landau correctly states that L should be the "least dimension of the system", an important distinction when dealing with the asymmetries from Lorentz contraction).

Landau also takes explicit note that he will be assuming zero viscosity and thermal conductivity, and again justifies this by simple dimensional arguments involving the Reynolds number R:

$$R = \frac{L\,V_{\text{BULK}}}{\lambda\,v_{\text{PARTICLE}}} \sim \frac{L}{\lambda} \gg 1$$

and since the "conventional" definition of Reynolds number in terms of viscosity η is

$$R = \frac{\rho\,L\,V_{\text{BULK}}}{\eta}$$

this limit is equivalent to assuming $\eta \to 0$. For much more on viscosity see my notes on [Viscosity done simply](#).

Thermodynamics Relations...

After the preliminaries, Landau works out the Bose and Fermi statistics for particles with arbitrary values of m/T. Most of this is simply expanding the relevant integrals for things like number density

$$n = g\,\frac{1}{(2\pi)^3}\int_0^\infty \frac{d^3p}{e^{(E-\mu)/T} \pm 1}$$

as a function of what he calls $z_\pi \equiv m_\pi/T$ and (for baryons, which are conserved and therefore have an associated chemical potential) $y \equiv \mu/T$. For our purposes, it's sufficient to just remind people of the basic blackbody results obtained in the $m = \mu = 0$ limit:

$$n_B(T) = g\,\frac{1}{(2\pi)^3}\int_0^\infty \frac{d^3p}{e^{p/T}-1} = g\,\frac{4\pi}{(2\pi)^3}\int_0^\infty \frac{p^2\,dp}{e^{p/T}-1}$$

$$= \frac{4\pi g}{(2\pi)^3}\,2\,\zeta(3)\,T^3 \approx g\,\frac{1.2}{\pi^2}\,T^3$$

This is the number density of a massless boson gas as a function of temperature; the function $\zeta(s)$ is the Riemann zeta function

$$\zeta(s) = \sum_{k=1}^\infty k^{-s}$$

. Taking $g=3$, we obtain the result Landau quotes between his Eqs. 2.4 and 2.5:

$$n_\pi(T) \approx 0.365\,T^3$$

.

In the end of this section, Landau derives a complicated expression relating the number of produced particles to the entropy. It's complicated only because he is solving the general case, in this massless limit for bosons it's straightforward. Let's first derive the total energy U of the gas:

$$U(T) = g\,\frac{V}{(2\pi)^3}\int_0^\infty \frac{p\,d^3p}{e^{p/T}-1} = g\,\frac{4\pi V}{(2\pi)^3}\int_0^\infty \frac{p^3\,dp}{e^{p/T}-1}$$

$$= g\,\frac{4\pi V}{(2\pi)^3}\,6\,\zeta(4)\,T^4 = g\,\frac{3V}{\pi^2}\,\frac{\pi^4}{90}\,T^4$$

.

Now use the general result $U = TS - PV + \mu N$ along with the special result $PV = U/3$ for a massless gas, and $\mu = 0$ by assumption to get

$$S = \frac{U + PV}{T} = \frac{4}{3}\frac{U}{T} = g\,\frac{4}{3}\,\frac{4\pi V}{(2\pi)^3}\,6\zeta(4)\,T^3$$

$$\Rightarrow \frac{S}{N} = \frac{g\,\frac{4}{3}\,\frac{4\pi V}{(2\pi)^3}\,6\zeta(4)\,T^3}{\frac{4\pi g V}{(2\pi)^3}\,2\zeta(3)\,T^3} = \frac{4}{\zeta(3)}\,\zeta(4) \approx 3.60$$

Landau tabulates the reciprical of this value in his Table 5; I'm a little puzzled why he lists the asymptotic value as 0.25 rather than the 1/3.6 = 0.28; I think it's because even for $\mu = 0$ he is still allowing for nucleon-antinucleon formation in his Eq. 2.20, and that results in a small correction.

The Total Number of Particles

The basic results of the previous section can be used to relate the total energy in the system to the number of produced particles. The fundamental assumption is that the expansion is isentropic ("proved" in next section). Let's write our relations for number density n, entropy density s and energy density ϵ as

$$n = aT^3 \,, \quad s = bT^3 \,, \quad \epsilon = cT^4.$$

Putting aside all the things that people later concluded were "wrong" about the Landau model (related to inability to put all the low-x partons into an extremely Lorentz-compressed volume), we would have for the initial (known) total energy , which then tells us immediately both the initial and final entropy, and thus the final number of particles:

$$S_i = b\,T_i^3\,V_i = b\left(\frac{U_i}{c\,V_i}\right)^{3/4} V_i = S_f = \frac{b}{a}\,N_f$$

$$\Rightarrow N_f = a\left(\frac{U_i}{c\,V_i}\right)^{3/4} V_i$$

Writing $U_i = A\,E_{CM}$, where A is the atomic mass, and assuming that the initial volume $V_i = V_N/\gamma = A\,V_0/(E_{CM}/2\,m_N)$, this becomes

$$N_f = a\left(\frac{U_i}{c\,V_i}\right)^{3/4} V_i = a\,A\left(\frac{2\,V_0^{1/3}\,m_N}{c}\right)^{3/4}\left(\frac{E_{CM}}{2\,m_N}\right)^{1/2}.$$

(Here m_N is the nucleon mass and V_0 is the "effective volume" of a nucleon.) Using the results from the previous section to substitute $a = g\,\dfrac{\zeta(3)}{\pi^2}$ and $c = g\,\dfrac{\pi^2}{30}$, and writing $V_0 = \dfrac{4\pi}{3}\,r_0^3$, this becomes

$$N_f = A\,g\,\frac{\zeta(3)}{\pi^2}\left(\frac{2\,(4\pi/3)^{1/3}\,r_0\,m_N}{g\,\frac{\pi^2}{30}}\right)^{3/4}\left(\frac{E_{CM}}{2\,m_N}\right)^{1/2}$$

$$= A\,g^{1/4}\,2.34\left(\frac{E_{CM}}{2\,m_N}\right)^{1/2} = 3.08\,A\left(\frac{E_{CM}}{2\,m_N}\right)^{1/2}$$

Putting in the values for full energy RHIC Au+Au collisions, I get

$$N_f = 3.08\,A\left(\frac{E_{CM}}{2\,m_N}\right)^{1/2} = 6260$$

which is certainly the right order-of-magnitude (note this includes π^0's, since I used g=3 above), and actually probably much better than an order-

of-magnitude.

We can check for (eventual) self-consistency by integrating the BRAHMS results that appear in the Introduction. These are gaussian in shape, with an intercept of about 300 pions per charge state, which (putting in a width of about 2.4) gives

$$\frac{dN}{dy} \approx \frac{300}{\sqrt{2\pi\sigma^2}} \, e^{-y^2/2\sigma^2} \Rightarrow N = 300\sqrt{2\pi\sigma^2} = 1800$$

This of course would give 5400 pions, as compared to the 6300 estimated above. However, the entropy counting is probably smarter than we are, since it we know the pions are only about 80(?)% of the total multiplicity, so that the 5400 pions implies a total multiplicity ~ 6750, so the "error" is within my own uncertainty on the 80% I am using... Amazing.

==WARNING==: This "agreement" should be taken with a very large grain of salt. Most practitioners today would say A) you need to take into account the inelasticity (~0.5) observed in p+p collisions, i.e., you can't take credit for the full energy being deposited in the initial volume and B) you also can't take credit for the full Lorentz contraction, since the uncertainty relation prohibits that degree of localization for partons below a given x value. I fully agree with these concerns, while at the same time noting they imply a next round of questions: a) Does the inelasticity in p+p matter if the system "immediately" becomes so dense that the forward-going energy in one collision is "contained" in subsequent ones and b) what fraction of the initial energy is carried by the low-x partons that extend "outside" the contracted volume?

Energy and Angle Distribution of Particles

That section title is of course Landau's; here I am mostly interested in understanding how the hydrodynamic equations of motion are applied and solved. That begins from

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = 0,$$

and writing the stress-energy tensor for a perfect fluid in terms of the energy density ε, the pressure P and the fluid four-velocity u as

$$T_{\text{pf}}{}^{\mu\nu} = (\epsilon + P)u^\mu u^\nu - P g^{\mu\nu}.$$

Note: I am using the Bjorken and Drell metric convention (1, -1, -1, -1), as appropriate for particle physics, and will try to be not too sloppy about upper and lower indices. This results in various minus signs differences, e.g., compare above to Landau's Eq. 4.2

Before proceeding with substituting $T_{\text{pf}}{}^{\mu\nu}$ into the equation of motion, it's useful to work out some thermodynamic relations related to the combination $\epsilon + P \equiv \omega = $ "heat function per unit volume". One more time, recall the general thermodynamic relations

$$E = TS - PV + \mu N$$

$$\text{with } dE = TdS - PdV + \mu dN$$

The "heat function" or *enthalpy* is defined by chemists as $H \equiv E + PV$, and Landau chooses to call the volume density of $H/V = \omega$. In the case where the number of particles is not conserved, we have $\mu = 0$, and life simplifies. In particular, we immediately have . Not so obvious, but true, is the all-important differential $d\epsilon = Tds$, which follows in a straightforward way from $d(E/V)$ and the above relations:

$$d\epsilon = d\left(\frac{E}{V}\right) = \frac{dE}{V} - E\frac{dV}{V^2} = \frac{TdS - PdV}{V} - (TS - PV)\frac{dV}{V^2}$$

$$= \frac{TdS}{V} - \frac{PdV}{V} - TS\frac{dV}{V^2} + P\frac{dV}{V} = Td\left(\frac{S}{V}\right) = Tds$$

A little later, we will also need the following:

$$dP = d\omega - d\epsilon = d(Ts) - Tds = sdT.$$

Now we're prepared to solve the hydrodynamic equations of motion for a perfect fluid. The basic approach is to project the four-vector of the equation of motion (Eq. 14) along and perpendicular to the four-velocity:

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\{\omega u^\mu u^\nu - P g^{\mu\nu}\}$$

$$= \frac{\partial \omega}{\partial x^\nu} u^\mu u^\nu + \omega \frac{\partial u^\mu}{\partial x^\nu} u^\nu + \omega u^\mu \frac{\partial u^\nu}{\partial x^\nu} - \frac{\partial P}{\partial x^\nu} g^{\mu\nu} = 0.$$

First project along the fluid four-velocity by contracting with u_μ and using $u_\mu u^\mu = 1$

$$\frac{\partial \omega}{\partial x^\nu} u^\nu + \omega u_\mu \frac{\partial u^\mu}{\partial x^\nu} u^\nu + \omega \frac{\partial u^\nu}{\partial x^\nu} - u^\nu \frac{\partial P}{\partial x^\nu} = 0.$$

The quantity that appears in the second term is in fact zero (yes, I am being slightly sloppy about lower and upper indices, but it's an instructive exercise to the student to verify why this works):

$$u_\mu \frac{\partial u^\mu}{\partial x^\nu} = \frac{1}{2} \frac{\partial (u_\mu u^\mu)}{\partial x^\nu} = 0,$$

so the equation becomes

$$\frac{\partial \omega}{\partial x^\nu} u^\nu + \omega \frac{\partial u^\nu}{\partial x^\nu} - u^\nu \frac{\partial P}{\partial x^\nu}$$

$$= \frac{\partial \epsilon}{\partial x^\nu} u^\nu + \frac{\partial P}{\partial x^\nu} u^\nu + (\epsilon + P) \frac{\partial u^\nu}{\partial x^\nu} - u^\nu \frac{\partial P}{\partial x^\nu}$$

$$= \frac{\partial \epsilon}{\partial x^\nu} u^\nu + (\epsilon + P) \frac{\partial u^\nu}{\partial x^\nu} = 0$$

Substituting in our thermodynamic relations, this beomes

$$\frac{\partial \epsilon}{\partial x^\nu} u^\nu + (\epsilon + P) \frac{\partial u^\nu}{\partial x^\nu} = \frac{T \partial s}{\partial x^\nu} u^\nu + T s \frac{\partial u^\nu}{\partial x^\nu} = T \frac{\partial (s u^\nu)}{\partial x^\nu} = 0$$

,

that is, the motion preserves the entropy, i.e., it's *adiabatic* (!).

We get the "other" equation of motion by projecting out the direction orthogonal to the four-velocity:

$$\frac{\partial T^{\mu\nu}}{\partial x^\nu} - u^\mu u_\lambda \frac{\partial T^{\lambda\nu}}{\partial x^\nu} = 0$$

.

(Note that this projection works just as for Cartesian vectors, i.e., for any vector A we find the part of it orthogonal to u by constructing $A - (A \cdot u) u$. All that's different here is that our vector A is produced by taking the divergence of the stress-energy tensor.)

Substituting the perfect fluid tensor, we obtain

$$\omega u^\nu \frac{\partial u^\mu}{\partial x^\nu} + u^\mu u_\lambda \frac{\partial P}{\partial x_\lambda} - \frac{\partial P}{\partial x_\mu} = 0$$

(after canceling many terms and again using $\partial_\mu(u_\lambda u^\lambda) = 0$).

At this point, it's worth noting that a) this is just the Euler equation for fluid motion, and b) the Euler equation is just $F=m\,a$. The "$m\,a$" term should be obvious above; the other two terms just project out the relevant component of the force along the fluid motion. See for example the discussion in *Gravitation*, Misner, Thorne and Wheeler, p. 562ff.)

Using the thermodynamic relations $\omega = T s,\ dP = s d T$, the Euler equation (Eq. 25) can be recast as

$$u^\nu \frac{\partial (T u^\mu)}{\partial x^\nu} - \frac{\partial T}{\partial x_\mu} = 0$$
.

Calculation details:

$$\omega u^\nu \frac{\partial u^\mu}{\partial x^\nu} + u^\mu u_\lambda \frac{\partial P}{\partial x_\lambda} - \frac{\partial P}{\partial x_\mu} = 0$$

$$T s u^\nu \frac{\partial u^\mu}{\partial x^\nu} + u^\mu u_\lambda s \frac{\partial T}{\partial x_\lambda} - s \frac{\partial T}{\partial x_\mu} = 0$$

$$T u^\nu \frac{\partial u^\mu}{\partial x^\nu} + u^\mu u_\nu \frac{\partial T}{\partial x_\nu} - \frac{\partial T}{\partial x_\mu} = u^\nu \frac{\partial (T u^\mu)}{\partial x^\nu} - \frac{\partial T}{\partial x_\mu} = 0$$

The assertion is then that this equation takes on a simple potential form in 1+1 dimensions. I am fairly certain this does not follow directly from the equation, one must also use the isentropic condition? and some more thermodynamic relations?):

$u^\mu = (u_t,\ u_z)$ only $\Rightarrow$

$$\frac{\partial (T\,u_t)}{\partial z} = -\frac{\partial (T\,u_z)}{\partial t}$$

$$\Rightarrow \exists\ \varphi \ni T\,u_t = \frac{\partial \varphi}{\partial z}\,,\quad T\,u_z = -\frac{\partial \varphi}{\partial t}$$

or $d\varphi(t,\ z) = -T\,u_t\,dz + T\,u_z\,dt$

A note on signs: once I start labeling space and u components with names "t and z" rather than indices (0,1,2,3) I am no longer regarding them as objects upon which we raise and lower indices-

they're just names for the functions $u_t(t,\ z)$ and $u_z(t,\ z)$ defined in the first equation line above.

Landau next effects two transformations on the above "potential" φ. Let's take these one at a time: First, plan on changing $\varphi(t,\ z)$ to a new function $\chi(T,\ u)$ via a Legendre transformation:

$$\varphi(t,\ z) \to \chi(T,\ u) \equiv \varphi + T\,u_t\,t - T\,u_z\,z$$

$$\Rightarrow d\chi = (\text{something})\,dT + (\text{something else})\,du$$

Note that it is allowed to write something like "du" in the 1+1 case, i.e., the u_t and u_z components are not independent. Make this explicit by parameterizing the four-velocity in terms of rapidity:

$u_t \equiv \cosh\alpha \Rightarrow u_z = \sinh\alpha$

then effect the Legendre transformation:

$$d\varphi(t, z) = -T\, u_t\, dz + T\, u_z\, dt = -T\,_{\cosh\alpha}\, dz + T\,_{\sinh\alpha}\, dt$$

$$\Rightarrow d\chi = d(\varphi + T\, u_t\, t - T\, u_z\, z) = t\, d(T\, u_t) - z\, d(T\, u_z)$$
$$= t\, d(T\cosh\alpha) - z\, d(T\sinh\alpha)$$
$$= (t\cosh\alpha - z\sinh\alpha)\, dT + (t\sinh\alpha - z\cosh\alpha)\, T\, d\alpha$$

Here I've kept the fairly crummy notation of using α for rapidity, since calling it the more standard "y" would probably just confuse things and would definitely increase the separation from Landau's notation.

. **To be continued(?)...**

Viscosity done simply

Here I provide some undergraduate material on viscosity. Almost all of this is on the "first coefficient", i.e., the one associated with *shear* viscosity. Other sources of dissipation are bulk ("second") viscosity and thermal conductivity.

The coefficient of viscosity is defined as

$$\frac{F}{A} = \eta\,\frac{d\,V_{BULK}}{d\,z}$$

which gives the force per unit area across two parallel surfaces when the fluid moving parallelt ot the surfaces has a velocity gradient $d\,V_{BULK}/d\,z$ between the two surfaces. This is a "drag" or dissipative force, and leads immediately to the concept of the Reynolds number as the dimensionless ratio of inertial to drag forces:

$$R = \frac{\text{momentum density}}{\text{momentum drag density}} = \frac{\rho\, V_{\text{BULK}}}{\frac{F\,d\,t}{A\,L}}$$

$$\sim \frac{\rho\, V_{\text{BULK}}}{\eta\, \frac{d\, V_{\text{BULK}}}{d\,z}\, \frac{dt}{L}} \sim \frac{\rho\, V_{\text{BULK}}}{\eta\, \frac{V_{\text{BULK}}}{L}\, \frac{L/V_{\text{BULK}}}{L}} = \frac{\rho\, V_{\text{BULK}}\, L}{\eta}$$

,

the last form being the conventional definition of the Reynolds number.

At the microscopic level, we understand viscosity as being due to the transport of momentum "across" some boundary due to thermal motion, so by dimensional considerations we would expect $\eta \sim \rho\, V_{\text{thermal}}\, \lambda$

$$\eta \sim \rho\, V_{\text{thermal}}\, \lambda$$

where λ is the mean free path. This immediately leads to the result that viscosity does not depend on the density(!):

$$\eta \sim \rho\, V_{\text{thermal}}\, \lambda \sim \rho\, V_{\text{thermal}}\, \frac{1}{\rho\,\sigma} \sim \frac{V_{\text{thermal}}}{\sigma}$$

and to the result Landau noted, that is, in the ideal hydro limit of large cross sections the viscosity goes to zero.

Much of the discussion of viscosity at RHIC has been in terms of the dimensionless ratio of viscosity to the entropy density s. Shuryak has asserted the flow data from RHIC suggest $\eta < 0.1 s$. What does this imply for the corresponding Reynolds number for the flow?

$$R = \frac{\epsilon\, v\, L}{\eta} \sim \frac{\epsilon\, v\, L}{0.1\, s} = 10\, \frac{\epsilon\, v\, L}{(\epsilon + P)/T} \sim 10\, \frac{\epsilon\, v\, L}{\frac{4}{3}\, \epsilon/T}$$

$$\sim 7.5\, v\, L\, T \sim 7\, L\, T \sim 7 \text{ to } 70$$

Here I've used the standard thermo results

$E = TS - PV + \mu N \Rightarrow s = (\epsilon + P)/T$ for μ=0. Writing $P = \epsilon/3$ is an ideal gas assumption; it's not at all clear to me how accurate this is for QGP, but I doubt it changes the qualitative result.

Curiously, these back of the envelope calculations put the Reynolds right in the transition regime, where vortices appear (say around R=20), up to where noise begins to appear (R=40), i.e., the flow pattern is no longer steady-state. (Feynman has a beautiful discussion of this in Volume 2 of his lectures.)

What would be the corresponding regime for water? The viscosity of water is about 10^{-3} Pa-S at room temperature. Let's assume we collide two 1mm drops of water. A collision velocity corresponding to a Reynolds number of 10 is then given by

$$v = \frac{R\eta}{\rho L} = \frac{10 \; 10^{-3} \, \mathrm{Pa} \, S}{10^3 \, \mathrm{kG}/m^3 \; 10^{-3} \, m} = \frac{0.01 \, m^2 \, \mathrm{Pa} \, S}{\mathrm{kG}}$$

$$= 10^{-2} \, \frac{m^2 \, N/m^2 \, S}{\mathrm{kG}} = 1 \, \mathrm{cm/sec}$$

This is a tricky analogy- we would not expect any great dissipative splat at these low velocities, but we must recall that the QGP flow apparently has the same smooth laminar structure even when compressed to the huge initial densities when elliptic flow is generated!

This is as good a place as any to calculate the ratio η/s for water. At STP, 1 mole of water has an entropy of about 70 J/K. This then gives an entropy density of

$$s = \frac{70 \, J/K}{k} \frac{1}{V} = \frac{70 \, J/K}{1.38 \; 10^{-23} \, J/K} \; \frac{1}{18 \; 10^{-6} \, m^3} = \left(\frac{2.81804 \times 10^{29}}{m^3} \right)$$

,

compared to a viscosity (which I convert to volume density using Planck's constant) of

$$\eta \sim 10^{-3}\,\text{Pa} - \sec = 10^{-3}\,\frac{J}{m^3} - \sec$$

$$= 10^{-3}\,\frac{J-s}{m^3}\,\frac{1}{(6.63\ 10^{-34}\,J-s\ /\ (2\pi))} = \left(\frac{9.4769\times 10^{30}}{m^3}\right)$$

leading to a ratio

$$\frac{\eta}{s} = \frac{9.4769\times 10^{30}\,m^{-3}}{2.81804\times 10^{29}\,m^{-3}} = 34$$

.

So, yes, I agree that the plasma value of ~0.1 for this ratio is much more "perfect" than water.

Somewhere people began describing the plasma as "sticky", perhaps as an attempt to explain its quenching properties. However, it's a very bad description of its *flow* properties. Our Reynolds number of 10 above corresponds to moving a 1 cm diameter stick through water at 1 mm/sec. I think most of us would agree that the stick would move "smoothly" through the water withour any sense of 'stickiness". By contrast, since the room temperature viscosity of honey is about 10^4 times larger than that of water, motion at the same Reynolds number in honey would require moving that stick at 10 m/sec (that's sticky!). Perhaps even more useful would be to think of moving the stick at the same velocity (1 mm/sec) as in water. This corresponds to Purcell's famous Life at Low Reynolds Number; of course you know intuitively that at this low Reynolds number the honey would indeed stick to the rod as you removed it from the fluid at 1 mm/sec.

Viscosity Done Not Simply

Perhaps the leading "not simply" candidate would be "Shear Viscosity in the O(N) Model", which after a nice introduction, becomes very opaque. But it

has a useful set of references, especially the rapidly-becoming-famous Policastro, Son, Starinets calculation in AdS. (As an aside, this 2001 paper notes that "the quark-gluon plasma one hopes to create in heavy-ion experiments has relatively low temperature at which the perturbation theory works very poorly".)

This later reference is readable because they make an effort to say in words the utterly bizarre connection they establish between viscosity on the gauge theory side and graviton absorption by a stack of N D3-branes. Here "N" is the same as that in the large-N limit, this because one wants to use the 't Hooft trick of letting N become large and working at "constant" $g^2 N$. The end result of this can be summarized as

$$\eta = f(g^2 N) N^2 T^3, \quad \text{where}$$

$$f(x) \to \frac{\pi}{8} \text{ for } x \gg 1, \quad x^{-2} \log^{-1}(1/x) \text{ for } x \ll 1$$

Is is the first limit that leads to the famous $\eta = s/(4\pi)$ that has the heavy ion world so excited. (The other limit is what you would get for a weakly coupled plasma, and I'm sure you can find it used in the literature...) They conclude "There is apparently no separation of scales in the strong coupling regime that would make a kinetic description possible: 1/T is the only length/time scale." I find it fascinating that you can find a discussion of precisely this connundrum by Kajantie in the proceedings of the *second* (1983) Quark Matter conference.

The leading reference for viscosity applied "directly" to observed RHIC data is Teaney's paper. However, as far as I can tell this is cast entirely as 1st order corrections to a Bjorken-scaling scenario (in fact, much of it is done as corrections to a blast-wave model, without reference to the EOS and/or underlying equations of motion). Very useful to get the lay of the land, but hard to extract quantitative limits on η from this. For instance, his result on the p_T dependence of v_2 is often cited as setting the limit on viscosity at RHIC:
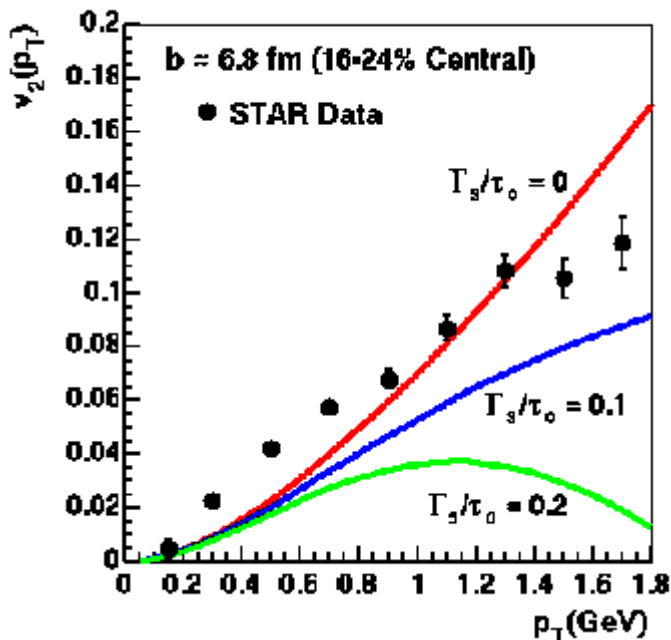
Here Γ_s is the sound attenuation length and τ_0 is some measure of the system size, so you can think of it as the reciprocal of the Reynolds number. The problem I see here is that the zero-viscosity (red) curve doesn't really describe the data all that well (wrong curvature), while turning on a little viscosity fixes the curvature, but misses the data... so what's the systematic error on these extractions??

Teaney is quite honest about inability to get a good global description of HBT data. Note also the reference to general theory of viscosity in relativistic fluids to S. Weinberg; you can also find a discussion of this in his *Gravitation and Cosmology* book.

Viscosity, Scott Pratt, and the Westinghouse

For obscure reasons, I posted an extensive discussion of viscosity to phenix-run-l some time ago; I reproduce it here for convenience:

```
Dear Colleagues:

This message concerns the relation between Scott Pratt's talk on QGP
signatures
and problems with the Westinghouse motor-generator set. It is for
educational
purposes only; no need to read past here unless you're interested in this
odd connection.

This follows a lunchtime discussion among several PHENIX people following
```

Scott's talk at the Users' Meeting, where he repeatedly used the terms
"viscosity" and "long mean free path" interchangeably. I think Scott's
point
was that ideal hydro describes much of the flow data, but it has no
viscosity,
but viscosity required long mfp's, but ... Hence the confusion.

Each of us involved in the lunchtime discussion (Soren Sorensen, Richard
Seto,
Glenn Young, myself) had some piece of the puzzle, but not the whole story.
(Perhaps the most interesting part of the discussion centered on Jason
Newby's
realization that these senior people did not exactly have a complete
knowledge
of hydrodynamics.)

Yes, the textbook formula for viscosity does has an explicit mfp in it.
The coefficient of viscosity η is defined in terms of the shear stress
(force per unit area) exerted on a slab by a velocity gradient in the
fluid:

$ F_x / A = \eta dv_x / dz $

Simple dimensional analysis suggests that (for an _ideal_ gas) that

$ \eta \sim   \rho v_{ave} L $,

that is, the product of the mass density, the average velocity, and the mfp
L.
This is also physically plausible, since it's basically the ability to
transport
momentum density ρv_{ave} over a distance L.

At the same time, it's puzzling. Doesn't hydro result in the limit of
short mfp's?
Does this mean hydro is incompatible with viscosity?? Of course not.
(Although many
of the hydro results in our field _are_ calculated ignoring viscosity, a
limit
Feynman refers to in not too favorable terms as "dry water" -- see lectures
40 and
41 in Volume II.) The "puzzle" is easily resolved when one expresses the
mfp in
terms of the cross-section $L = 1 /(n \sigma)$, where n is the number
density,
which makes it clear that the viscosity is *density independent*, i.e.,

$ \eta \sim   M v_{ave} / sigma $.

So there is *no* direct relation between "long" mfp's and viscosity, even
in the
ideal gas limit.

All of this is very well known and readily obtainable on the web, e.g., see

   http://www.chem.hope.edu/~polik/Chem345-
1997/gasviscosity/GasViscosity.html

for a nice discussion (but note usage of "Poiseville" where I think one
wants "Poiseuille")

and

http://www.science.uwaterloo.ca/physics/p13news/maxall.html

for the historical background (Maxwell was not only the first to derive the fact
that viscosity for gases is essentially pressure independent, but also conducted
experiments, together with Mrs. Maxwell, to verify the temperature dependence
implicit in the v_{ave}.)

Now clearly ideal gases don't behave as the viscous fluids such as honey or oil
we are all familiar with. There, the viscosity *decreases* with temperature, rather
than increasing as suggested by the result for ideal gases. Not surprisingly, this is
related to the fact that oil is not an ideal gas, or even an "ideal fluid".
In particular, there is an additional source of viscosity that results
from attractive forces between molecules, which has a strong temperature dependence,
i.e., higher temperatures prevent bonding and reduce internal "friction" in the fluid.

This last point is directly related (I am _guessing_) to the inability to turn on
the motor-generator set under high temperatures, as noted in Vicki Greene's shift
change message

http://www.phenix.bnl.gov/phenix/WWW/p/lists/phenix-run-l/msg02932.html

from 10-Aug-01:

>
> The problem with the Westinghouse yesterday was that the oil, which
> normally circulates through the cooling system, became warm during the
> shutdown. This caused the oil pressure to drop below the point where the
> W. would turn on. Some fixes were discussed, such as welding cooling fins
> onto the oil tank or replacing the 200 gallons of oil with all-weather,
> synthetic oil.   Where is the nearest Jiffy Lube?
>

A standard calculation for laminar flow of a viscous fluid (oil in this case)
in a pipe of diameter R shows that the pressure δP required to support
a given flow rate varies as

$ \delta P \sim \eta S / R^4 $

My guess is that the M-G set uses this as a way of measuring the oil pressure.
But for sufficiently hot oil, η is reduced, and the oil pressure fails to
reach the minimum value required to satisfy the interlock.

Best regards,

Bill

Stress-Energy Tensor Basics

To be completed...

Production Notes

To Do List:
1. References to modern viscosity papers (update Teaney wrt jets)
2. State of the art hydro, Hirano QM talk?
3. Stress-energy tensor basics

Another note-to-self: to convert this to pdf while preserving html tags, need to do (sigh): nb-->html-->edit with Word-->Distill Word file. See this link.